Asset Pricing

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‘General’ Fundamental Theorem of Finance

Markets are in equilibrium → No Arbitrage → Strictly positive probabilities

This state is never actually reached, as we will see, but from these assumptions we can calculate what prices *should* be

Arbitrage

Weak: An arbitrage opportunity exists if a positive payoff is possible in at least one future state of nature, with no initial investment and no possible losses

Strong: when one can form a portfolio that has a positive or zero payment tomorrow at every state but gives a positive payment today

Arbitrage Example

- Example: Unilever is traded at two stock exchanges (UK & Netherlands). It is the same stock ofc. but sometimes it might be a bit cheaper in one location (say the UK). So the trader shorts the expensive version (Netherlands) and buys the cheaper version (UK). He uses the UK stock to offset losses (or gains) from the Netherlands. No matter what happens, there will be zero total payoffs from the holding. He has invested nothing, and even has money left from the short to stuff his own pockets.
- This is called pairs trading and is a common high frequency trading strategy

Why no arbitrage

- Arbitrage is a way to make riskless, ‘free’ money
- Traders will take advantage of arbitrage opportunities
- By taking advantage, they will move prices
- If enough traders jump on the opportunity, it will disappear
- The traders are thus enforcing the law of one price (LOOP)

Law of one price

Securities with the same payoff profile should have the same price

E.g. Unilever in the UK and Netherlands should cost the same

This also holds for other stock, bond, option, etc. combinations that yield the same payoff

Constructing arbitrary payoff profiles

- By combining securities which have different payoff profiles for different states we can create new payoff profiles.
- This is in effect the same as creating linear combinations of payoff vectors of different assets.
- To figure out if we can create *any arbitrary* payoff profile, we have to calculate the span of the matrix containing the payoff profiles of all assets we can trade

The asset span

- Construct a (J,S) matrix X containing the payoffs of J securities at all states S
- Securities form rows with their payoffs in all states (potential outcomes)
- States form columns with the payoffs of all securities if a state actually occurs
- If the resulting matrix is full rank (see linear algebra class) markets are **complete**, that is, we can construct any payoff profile from them
- Otherwise, the matrix is **incomplete**, either because there are not enough assets or because assets are redundant, that is they are just linear combinations of other assets

Arrow Securities

- If we have a complete market, we can construct portfolios with arbitrary payoff profiles
- That means we can construct a security that pays 1 if a certain state s occurs
- We can do this for all states
- The resulting payoff matrix of those portfolios (called Arrow Securities) is the identity matrix

- The payoff matrix of an Arrow security looks like this: $\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$
- It is constructed by finding a set of holdings that lead to a payoff of 1 in each state and zero otherwise

$$8 = 1 \cdot q_1 + 2 \cdot q_2 + 3 \cdot q_3$$

$$4 = 1 \cdot q_1 + 1 \cdot q_2 + 1 \cdot q_3$$

$$5 = 1 \cdot q_1 + 1 \cdot q_2 + 2 \cdot q_3$$

$$\Rightarrow (q_1, q_2, q_3) = (1, 2, 1)$$

State pricing

- The price for an arrow security is the price we are willing to pay to receive 1 in if a certain state s occurs, thus they are called state prices
- We can find the state prices by solving $p = Xq$ where p are the prices of the securities traded, X is their payoff matrix and q are the state prices
- **This assumes that agents do not want to be compensated for the risk they are taking. State prices are risk neutral**

State prices and arbitrage

- If there was a negative state price, we could buy it now, *receive money now* and, in the worst case loose nothing later and in the best case receive even more money later.
- This is a form of strong arbitrage
- Thus, if there is to be no arbitrage, all state prices have to be positive
- Yet, state prices can be higher than one, investors might be willing to accept negative returns

State prices and probability (L2S21)

- An arrow security with a state price q pays 1 of the state occurs
- Thus arrow securities with higher probabilities of their state occurring demand higher prices to keep expected returns equal
- To obtain the probability of a state occurring π_s we have to normalize the state prices so that all probabilities sum to 1
- $\pi_s = \frac{q_s}{\Sigma q}$
- Since state prices are risk neutral, these probabilities are also called **risk-neutral probabilities**

Martingale probabilities (L2S28)

- The martingale property ensures that in a ”neutral” world, knowledge of the past will be of no use in predicting future. Only the information available today is relevant to make a prediction on future prices
- Formally: $P(x_{t+1}|x_1, \dots, x_n) = x_n$
- That is, the conditional expected value of the next observation, given all the past observations, is equal to the most recent observation
- A risk neutral and unconditional probability is called a Martingale probability
- If knowledge of the past could lead to different probabilities, it would give rise to arbitrage opportunities. Thus, for no arbitrage all state probabilities π_s need to be Martingale probabilities (or Martingale measures)

Dynamic completion (L2S27)

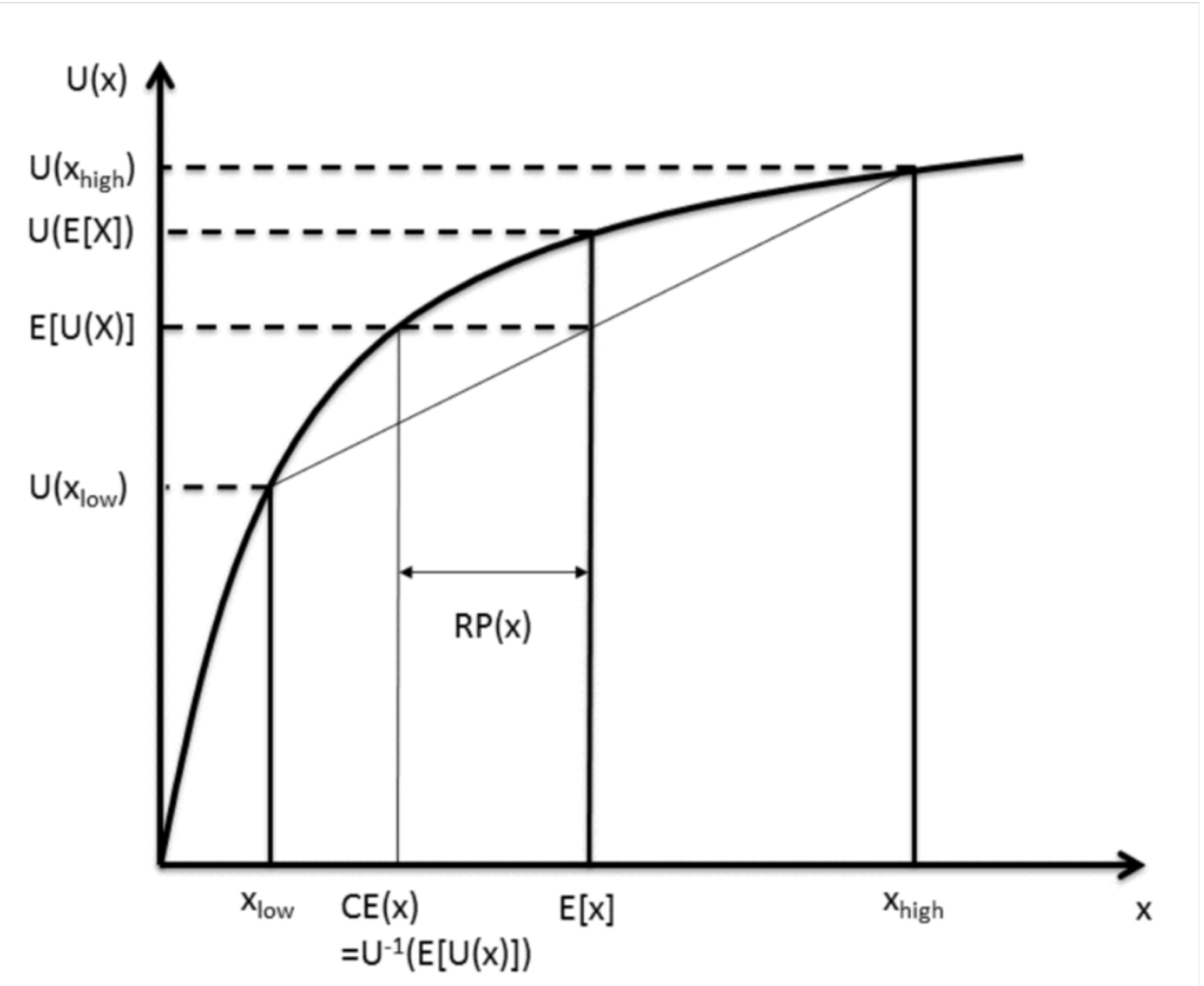
- Even if the market is incomplete we might be able to solve for the Martingale measures and arrive at one (unique) solution
- If we know the Martingale measures of all states we can price any security knowing the payoff for each state, *even though markets are incomplete*
- This process is called **dynamic completion**
- For dynamic completion of markets the number of securities traded must be no less than the maximum number of branches emanating from each node on the event tree
- For a good example see slide 29 of the second lecture

Risk aversion

- Most people are **risk averse**, just *how* risk averse depends on the person
- People with a higher risk aversion will demand a higher **risk premium**
- People who are **risk neutral** will demand no risk premium
- Traders value assets with a **risk neutral valuation**. If assets were priced under risk preference considerations, it would give rise to arbitrage opportunities
- An agent might still demand a risk premium if he can get it on the market

Certainty equivalents (L2S10)

- Imagine you got offered a 50% chance of \$2M (expected gain \$1M) or \$X for sure. The amount X has to be for you to be indifferent to the risk and the sure thing is called the certainty equivalent (CE) to the expected \$1M
- For risk averse agents the curve of risky expectation of CE is concave as they demand a risk premium when gains are risky



Risk aversion and utility (L2S12)

- A rephrase of the last slide is that the expected utility of some fixed gain y plus risky gains (or losses) z must be equal to the utility from the fixed gain y minus the risk premium $\rho(y, z)$
- $\mathbb{E}[U(y+z)] \; = \; U(y-\rho(y,z))$
- Where $y-\rho(y,z)$ is the certainty equivalent of the risky consumption plan $y+z$

Absolute and relative risk aversion

- As you can see, the agents risk premium and risk aversion depends on the agents Utility function.
- The absolute risk aversion (ARA) is:
- $A(W) = \frac{U''(W)}{U'(W)}$ In which the agent can loose a fixed amount and his aversion to this risk is dependent on his total wealth
- The relative risk aversion (RR) is:
- R$(W) = -W\frac{U''(W)}{U'(W)}$ In which the agent can loose a fixed share of his wealth and his aversion to this risk is dependent on his total wealth

Arbitrage Pricing Theory (L2S36)

- Many application require a non-risk-neutral valuation
- The main tool for such valuation is CAPM
- APT achieves the same outcome but using arbitrage arguments
- In APT, there is a stable set of factors that explain all returns of all assets
- Primitive securities work similar to arrow securities as their returns and thus risk are only determined by one factor

APT II

- The return of a security j is thus determined as:
- $r_j = a_j + b_{1j} f_1 + \cdots + b_{kj} f_k + e_j$
- Where a_j is the offset, b_{kj} is the 'loading factor' of security j with respect to factor k, e_j is some mean zero error
- In expectation this evaluates to:
- $\mathbb{E}(r_j) = r_f + \beta_{1j}(\mathbb{E}\left(r_{f_1}\right) - r_{f_1}) + \cdots + \beta_{kj}(\mathbb{E}\left(r_{f_k}\right) - r_{f_k})$
- β_{kj} is the correlation between the security returns and the factor returns

APT and CAPM (L2S41)

- If there is only one factor, and that factor is the market, APT collapses to
- $\mathbb{E}(r_j) = r_f + \beta_j(\mathbb{E}(r_m) - r_m)$
- Which is the same as CAPM.
- There are reasons CAPM only assumes one factor, not only because it is much easier to compute. But recently the idea of using more factors has gained traction
- See https://tsvenn.com/ for an example

Options (L3S7)

- A **call** option is the right to **buy** an asset at a pre-specified price K
- A **put** option is the right to **sell** an asset at a pre-specified price K
- A **European option** gives this right only at maturity of the option
- An **American option** gives this right any time before or at maturity
- An **Asian option** works like a European option, only that K is fixed to be the average price of the asset during the options lifetime
- European options are the easiest to model which is why most theory is about them

Put Call Parity (L3S14)

- For a European call option with price c and put option with price p (on the same asset with the same strike price K):
- $c + \frac{K}{1+r_f} = p + S_0$
- This is because the two portfolios on each side of the equation have the same payoff profile

Completing markets with options (L3S15)

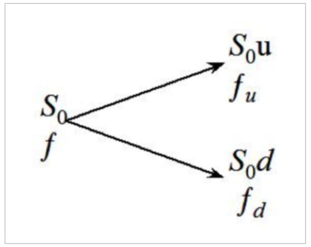
- Options can complete markets.
- As long as the asset has different payoffs for all states we can construct a range of new securities in which the payoff at any number of states is zero, but not less.
- Because the option payoff shape is not linear, these new securities are not linear combinations of each other or the original asset.
- If an asset has the same payoff at different states, we can not create an option that is in the money in one state but not in the other.

American options (L3S19)

- Because American options let an investor do more, they are at least as much as an European option.
- Via European put-call parity we can prove that an American **call** option **should never be exercises early** (for non dividend paying stocks)
- An American **put** option might be worth being exercised early, if the risk neutral probability times the discounted cash from a later exercise is lower than the current payoff
- In practice traders might exercise deeply in the money options to take the cash home, and ofc. there are dividends

Pricing an option in a two state model

- Assuming a two period economy in which the second period has two states, either the option value goes up or it to zero
- We can create a risk free portfolio by buying Δ stocks and short selling one option
- Δ has to be chosen so that gains and losses from the stock offset gains and losses from the option, the portfolio return should always be the risk free rate
- Thus Δ needs to be the ratio of the change in option price f over the change in stock price S
- $\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$



Options and Probabilities (L3S26)

- The value of the option should be its payoffs weighted by the risk neutral probability of those payoffs occurring discounted by the risk free rate
- $f = \frac{1}{1+r_f}[\pi^* f_u + (1-\pi^*)f_d]$
- We can thus substitute π^* into our previously derived formula for Δ
- $\pi^* = \frac{1+r_f-d}{u-d}$
- Where u and d are the up and down **1 + percentage change** of the underlying stock (see slide 29)
- Thus we can use options to calculate the risk neutral probabilities
- This is closely linked to completing the market with options

When to use binominal tree approaches (L3S46)

- If the option can be exercised early (e.g. American options)…
- When dividends are paid…
- When the strike price is computed from movement of the underlying (e.g. Asian or lookback options)
- If the payoff depends on another option (Compound options)

Cox-Ross-Rubinstein option pricing (L3S32)

- The value of an option is determined by the number of its up moves, and the probability of each up move is the risk neutral probability $\pi^* = \pi_u$
- We can thus model the probability of obtaining n up moves over T periods using a binominal distribution
- Effectively the price of an option over T periods is the value of the option if k of those periods go up, times the probability of the option going up k many times for all values k from 0 to T
- $c = R^{-T} \sum_{k=0}^{T} \binom{T}{k} \pi_u^k \pi_d^{T-k} \, [u^k d^{4-k} S - K]^+$
- Where R = 1+rf, $\pi_d = (1 - \pi_u)$, S being the initial stock price and u and d are the relative up and down movements

The Black-Scholes Model (L3S40)

- Black-Scholes, sometimes Black-Scholes-Merton or BSM model
- Khan academy has a good overview: https://youtu.be/pr-u4LCFYEY
- Assumes continuous time, instant trading
- Assumes a constant risk free rate
- Assumes no transaction costs
- Assumes stock movements are normally distributed (Brownian motion), with constant drift and volatility
- Assumes short selling comes at no extra cost

BSM II (L7S37)

- The value of a call option c with strike price K and initial price S_0 and a run time of T is:
- $c = S_0 N(d_1) - K e^{rT} N(d_2)$
- Where
- $d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$
- $d_2 = d_1 - \sigma\sqrt{T}$

- If the volatility σ of the underlying goes up, d_1, $N(d_1)$ and thus c go up
- If the volatility of the underlying goes down, d_2, $N(d_2)$ go up and c goes down
- Thus, **an option is always a bet on volatility**
- From the option price **we can calculate the implied volatility of the underlying**

$$c = \underbrace{S_0 \cdot N(d_1)}_{PV[\mathbb{E}(C_T^1)]} \underbrace{-K \cdot e^{-rT} \cdot N(d_2)}_{PV[\mathbb{E}(C_T^2)]}$$

$$C_T^1 = \begin{cases} S_T & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

We gain the stock if the price is over the strike price

$$C_T^2 = \begin{cases} -K & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

We have to pay the strike price if it is below the stock value

- The value of an option is the value of the exercised option exercised minus the value of not-exercised option times the probabilities of exercising $N(d_1)$ and not exercising $N(d_2)$ respectively.
- The probability of exercising the option rises with increased volatility

BSM III (L7S28)

Merton's model (L7S34)

- Equity can be seen as an option on the company's assets with the strike price equal to the total debt of the company
- As a shareholder (owning the entire company), you could payoff the debt, dissolve the company and take the assets
- Under this model, it makes sense of an investor to exercise the option if the companies assets are worth more than its debt
- The investor would **not** exercise the option if the company is in default (Assets < Debt)
- We can use Black-Scholes to estimate the implied probability of credit default from the stock price

Merton's model and BSM

- To model equity as an option on assets, the strike price becomes the total debt D, the initial stock price becomes the initial asset value V_0, the volatility of the underlying becomes the volatility of assets $\boldsymbol{\sigma}_V$ and the value of a call option becomes the value of equity at time zero E_0
- $E_0 = V_0 N(d_1) - De^{rT} N(d_2)$
- $d_1 = \frac{\ln\left(\frac{V_0}{D}\right) + \left(r + \frac{\sigma_V^2}{2}\right)T}{\sigma_V \sqrt{T}}$
- $d_2 = d_1 - \sigma_V \sqrt{T}$

Merton's model and BSM II

- The value of the option can be seen as the weighted gains from the option, with not paying the strike price being a gain so: $1 - N(d_2) = N(-d_2)$ expresses the probability of an investor not exercising the option and equals the probability of default
- The recovery rate (how much of the loan we can expect to recover from default) is given as:
- $RR = \frac{V_T}{D} e^{rT} \frac{N(-d_1)}{N(-d_2)}$
- Value and volatility of assets are estimate from equity movements
- $\sigma_E E_0 = N(d_1)\sigma_V V_0$
- Thus, tadaaa, we can estimate probability of default and recovery rate from equity movements

Notes on Merton’s Model (L7S43)

- The model makes a number of assumptions including: No taxes, transaction costs, bankruptcy costs, stock repurchases, issuing of new debt
- By summing up the assumed payoff schemes for debt and equity holders, we can show that no matter the leverage, the value of the company stays constant, the famous Modigliani-Miller theorem
- d_2 from the model is often taken alone as a ‘distance to default metric’
- There are numerous extensions and several services of credit risk ratings are built on this model

Fixed income options (L6S27)

- Options are used in fixed income to create a floor or a cap on interest payments
- If an entity has to pay an interest rate based on LIBOR, it can buy a call option on LIBOR (a caplet). If LIBOR rises, gains from the option will offset the higher interest costs, effectively capping interest payments
- To cap interest payments of a loan over a longer period of time, a series of caplets has to be bought and combined into a cap
- If a lender wants to floor interest rates, it can buy one or a series of puts on LIBOR which will offset losses from a falling LIBOR and floor revenues from the loan
- A cap and a floor together can be used to fix an interest rate, which is what a Swap does, so
- **Price of Cap – Price of Floor = Price of Swap**

Swaption (L6S31)

- A swap is a contract in which the **payer** pays the **receiver** a fixed interest rate K and the receiver pays a floating LIBOR rate back. The fixed / floating exchange rate is called **swap rate**
- A **receiver swaption** is an option for the receiver to enter a swap in which it receives a pre arrange fixed rate K in return for floating rates
- A **payer swaption** is an option for the receiver to enter a swap in which it pays a pre arrange fixed rate K in return for floating rates
- A receiver swaption is thus in effect a put option on the swap rate and a payer option a call on the swap rate
- Price of receiver swaption – price of payer swaption = Forward swap rate

Swaptions vs Caps (L6S38)

- A cap allows you to draw or not draw the caplet at any time t
- Thus, you are always on the better side of being in a swap or not
- A swaption on the other hands means that once you are in, you are in
- Thus, a cap has higher payoffs and is more expensive than a swaption

Equilibrium

So far, we have priced all things with **no arbitrage** arguments

The other way is to price under **equilibrium** arguments

Remember, that under the fundamental financial theorem, no arbitrage means there is an equilibrium, too

What is an market equilibrium (L4S18)

- An equilibrium is defined as a state in which each agent maximizes its utility, subject to the constraint that…
- Initial Wealth minus consumption ≥ invested capital
- Wealth minus consumption ≥ payoffs from investment
- Consumption can not be larger than wealth
- Total asset holdings of an asset must sum to zero
- For a formal definition see slide 18

Pareto Optimality (L4S21)

- Assets are distributed in a way, that *no one can get better off without someone else getting worse off*
- Sufficient condition of Pareto Optimality: The marginal rates of substitution (MRS) across states of the world are equal for all agents
- If the world is Pareto Optimal, no trade occurs
- When markets are complete, a financial equilibrium is Pareto Optimal
- When markets are incomplete, a financial equilibrium is only Pareto Optimal for one asset

An Arrow-Debreu Economy (L4S5)

- A two period economy of each L commodities and I investors
- Investors trade ‘contingent commodities’, a commodity whose delivery is conditional on one certain state occurring
- Contingent commodities are like Arrow securities, but with different commodities instead of only cash. They materialize in the second period, but investors can trade the security with a chance to get them in the first period
- Next to having contingent commodities, investors gain a fixed amount of w commodities each period and aim to consume c commodities to maximize their utility

Arrow-Debreu equilibrium (L4S9)

- For an equilibrium to occur, 3 conditions must be met
- 1. Investors can not consume more than they own
- 2. They must choose a set of commodities that maximizes their utility
- 3. All commodities must be consumed, supply must equal demand
- If these are true then the price of a contingent commodity ls is the ratio between the extra utility from having (consuming) more ls over the extra utility of having more money
- Formally: $\frac{q_{ls}}{q_{10}} = \frac{u'^i(c^i_{ls})}{u'^i(c^i_{10})}, \forall s \in S$, q_{10} stands for the price of money

AD-Economy & Market Equilibria (L4S23)

- The financial market equilibrium with complete markets collapses to the Arrow-Debreu equilibrium
- If markets are complete we can replace the payoff matrix with arrow securities and thus security prices with state prices.
- The securities are now equivalent with contingent commodities from the AD-Economy

Asset Markets model (L4S12)

- Similar to the AD-Economy, a two state economy, BUT
- Consumers face a multiplicity of budget constraints, at different times and under different states of nature
- To transfer wealth among budget constraints (and not just time as in the AD-Economy), consumers must hold assets
- Re-trading at t=1 is important and agents have to correctly anticipate today the price that will prevail tomorrow

AM-Model II (L4S15)

- By optimizing with Kuhn-Tucker methodology we arrive at a value for prices
- Prices equal payoff times derivative of utility with respect of consumption in later state divided by derivative of utility with respect to consumption in initial state
- $p = X\frac{\frac{\partial u}{\partial c_1}}{\frac{\partial u}{\partial x c_0}}$
- The extra utility of having the asset later over the extra utility of having the asset now (usually a fraction < 1) times the amount of payoff we will get

Walras Law (L4S16)

The excess demand is consumption minus wealth

The price weighted excess demand must be zero

Stochastic discount factor (L4S33)

- $\frac{\frac{\partial u}{\partial c_1}}{\frac{\partial u}{\partial x c_0}}$, or $\frac{u'(c_{1s})}{u'(c_0)}$ with an extra s if the economy has multiple states
- Called: **marginal intertemporal rate of substitution between today and state s tomorrow**
- Also known as **pricing kernel** or **stochastic discount factor (SDF)**
- Means that the value of a security does not only depend on its payoff but of the extra utility of having them in the state they occur

Lucas Tree (rep. Agent) Model (L4S27)

- A two period economy but multiple possible states in the second period
- Each agent is endowed with a number of commodity producing trees as well as an amount of the commodity
- All agents are the same (utility function, endowment, etc.)
- Every agent maximizes current utility plus, expected future utility discounted by the factor δ
- In state zero, every agent may choose to sell a share $\boldsymbol{\phi}$ of its trees for the commodity

Lucas Tree Budget (L2S29)

- An agent faces two budget constraints
- In t=0, it can only consume the money you got plus the price of a tree times the trees sold
- In the future, it can only consume what your remaining trees give
- The price of a tree equals δ times the expected payoffs times the stochastic discount factor
- $p = \delta\mathbb{E}[\frac{u'(c_{1s})}{u'(c_0)}X]$, known as the **Euler equation**

Lucas Equilibrium

- Since all agents are the same, they would all want to buy or sell, so they won't find trade partners, so no trade will happen in t=0. Agents will consume their commodity endowments and then live off the trees
- Since no trade means markets are in equilibrium, our price P is an equilibrium result
- If the future discount factor δ is one, we can reformulate the equilibrium price to be the state price.
- $q_s = \pi_s \frac{u'(c_{1s})}{u'(c_0)}$, π_s is the subjective probability of state s occuring

Lucas and Insurance

- $u'(c_{1s})$ depends on income from trees and other income from wealth
- An arrow security is effectively an insurance to receive something in state s
- The lower the income in state s, the higher the price of insurance for state s

Breeden’s Formula (L4S36)

$$p_j = \overbrace{\frac{\mathbb{E}\left[X_j\right]}{R_f}}^{1.} + \overbrace{\frac{1}{R_f}\frac{Cov(u'(c_1), X_j)}{\mathbb{E}\left[u'(c_1)\right]}}^{2.}$$

- If a riskless asset pays 1 in every state then the payoff matrix X=1
- Thus the price of that asset from the Euler equation must be the same as 1 discounted by the risk free rate
- We can thus re-arrange the Euler equation where:
- **1.** Is expected income discounted by the risk free rate
- **2.** A risk premium

From Breeden to CAPM

- Assuming a linear consumption function whose parameters do not matter as long as it is linear we can derive CAPM from utility considerations
- For the fancy math see slide 36

Equity premium puzzle (L4S39)

- Many economic models assume a utility function named CRRA
- $u(c) = \frac{c-\rho}{1-\rho}$
- Where ρ is a risk aversion factor, usually between 1 and 4
- Plugging this utility function into Euler's equation yields possible equity prices compared to bond prices
- We can also try to estimate peoples risk aversion from real prices
- We find that people would have to be **very** risk averse (ρ in the hundreds or thousands)
- Thus, our models seem to either underestimate risk aversion or there is some other factor depressing equity prices

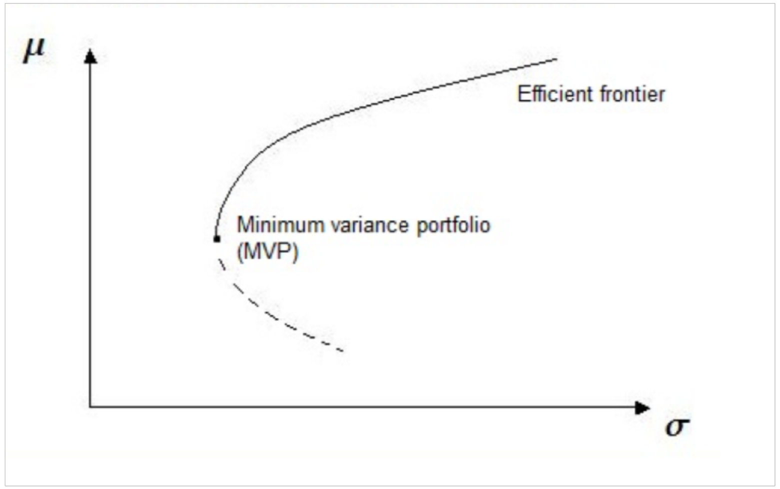
Deriving CAPM (L5S9)

- Assuming agents maximize the expected utility from their wealth at some end period
- They maximize the expected future return which is risk free returns plus some extra returns in risky assets
- Assuming a quadratic utility function, agents will prefer high mean returns and a low variance of returns (see slide 11)

Adding portfolio standard deviations (L4S12)

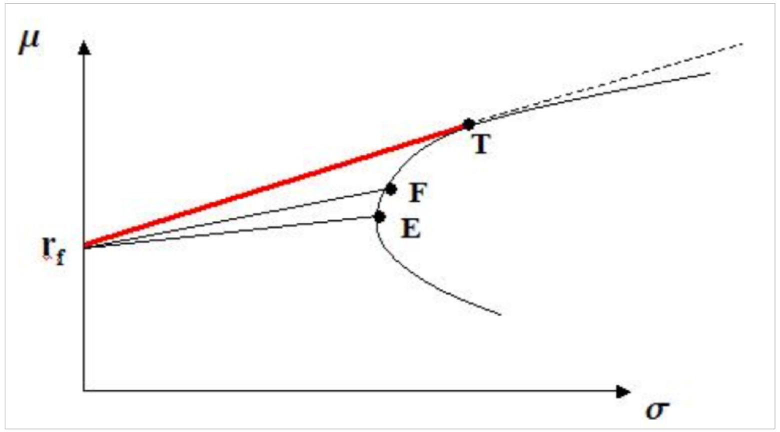
- To find an optimal portfolio, we need to sum the standard deviation of two assets with weights w
- Adding standard deviations work as usual, only weights need to be added:
- $\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$
- For two assets $w_1 + w_2 = 1$ so $w_2 = 1 - w_1$
- The upper bound of the joint standard deviation is the standard deviation at perfect correlation which simplifies to
- $\sigma_p^2 \leq (w_1\sigma_1 + (1 - w_1)\sigma_2)^2$
- The standard deviation of the portfolio returns is lower than the weighted average due to gains from diversification

Constructing the efficient frontier



- Given that the return of a portfolio is the weighted average return of its assets, and the standard deviation is the added standard deviation given previously, we can solve for a return at any given standard deviation and vice versa
- Note that this gives usually two solutions, the upper and lower side of the efficient frontier, separated by the minimum variance portfolio (MVP)
- We discard the lower side as it just delivers worse results

From portfolio to CAPM (L4S22)



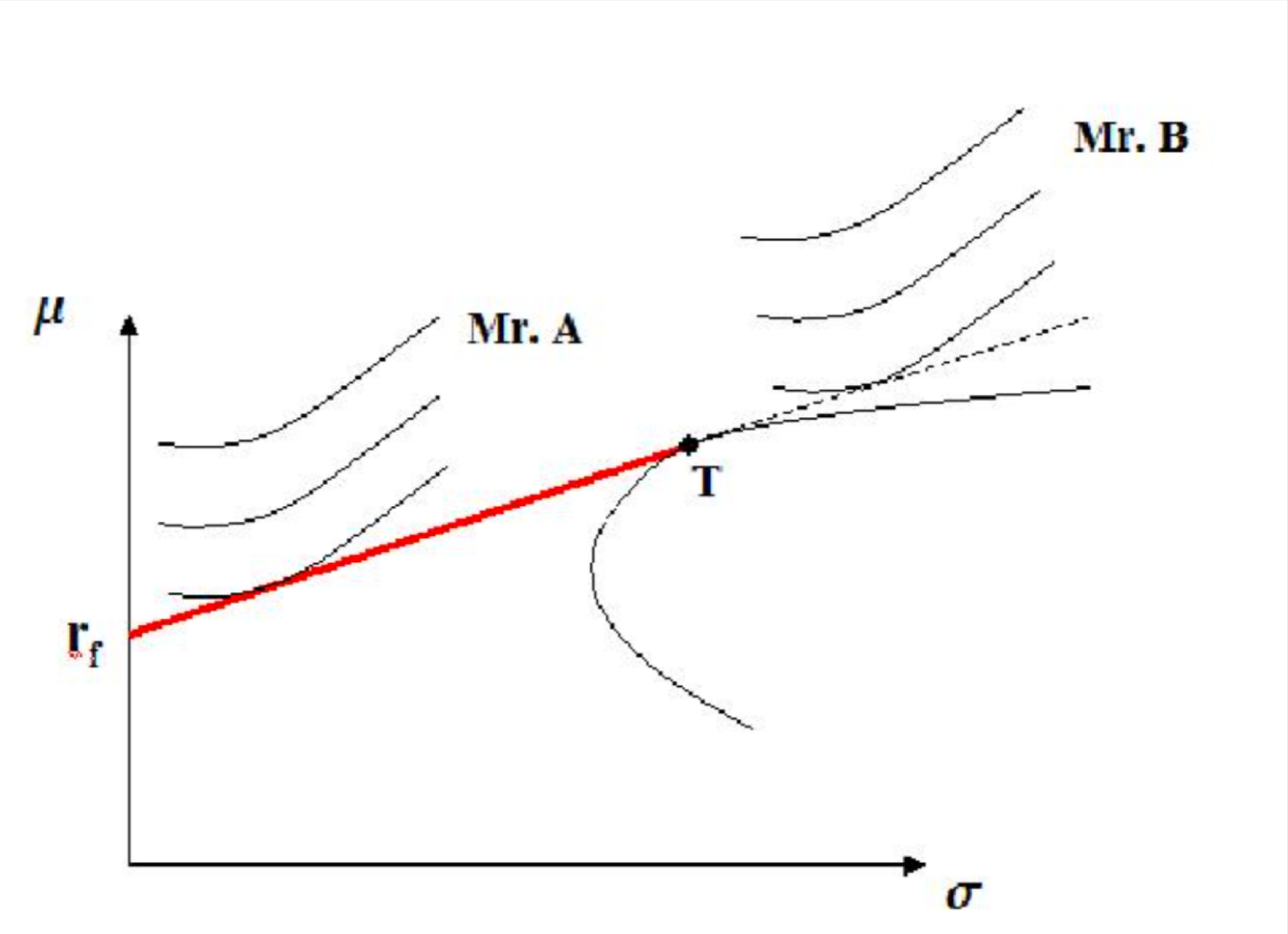
- We can combine a risk free asset with a risky asset
- $\mu_p = r_f + \frac{r_T - r_f}{r_T}\sigma_p$
- If we construct a portfolio with n risky assets and one risk free asset, the efficient frontier will always be a line from the risk free asset to a **tangent portfolio** of the risky assets
- The line from r_f through T is called the **capital market line (CML)**

The Sharpe ratio

- The slope of the efficient frontier, $\frac{r_T - r_f}{r_T}$, is called the Sharpe ratio
- It gives the price of risk, how much compensation one can expect for more risk
- If a portfolio has a Sharpe ratio higher than the market Sharpe ratio, it mean that an agent is getting more for the risk it takes and invests more efficient than everyone else. Which is why hedge funds are so hyped about the Sharpe ratio.
- In an efficient, equilibrium market of rational agents, all Sharpe ratios should be the same

A Separation Theorem (L4S27)

- If n includes all risky and one risk free assets, the efficient frontier is the best investment for **everyone**
- Agents with different risk preferences will just pick different points on the line, but all holding some combination of r_f and T



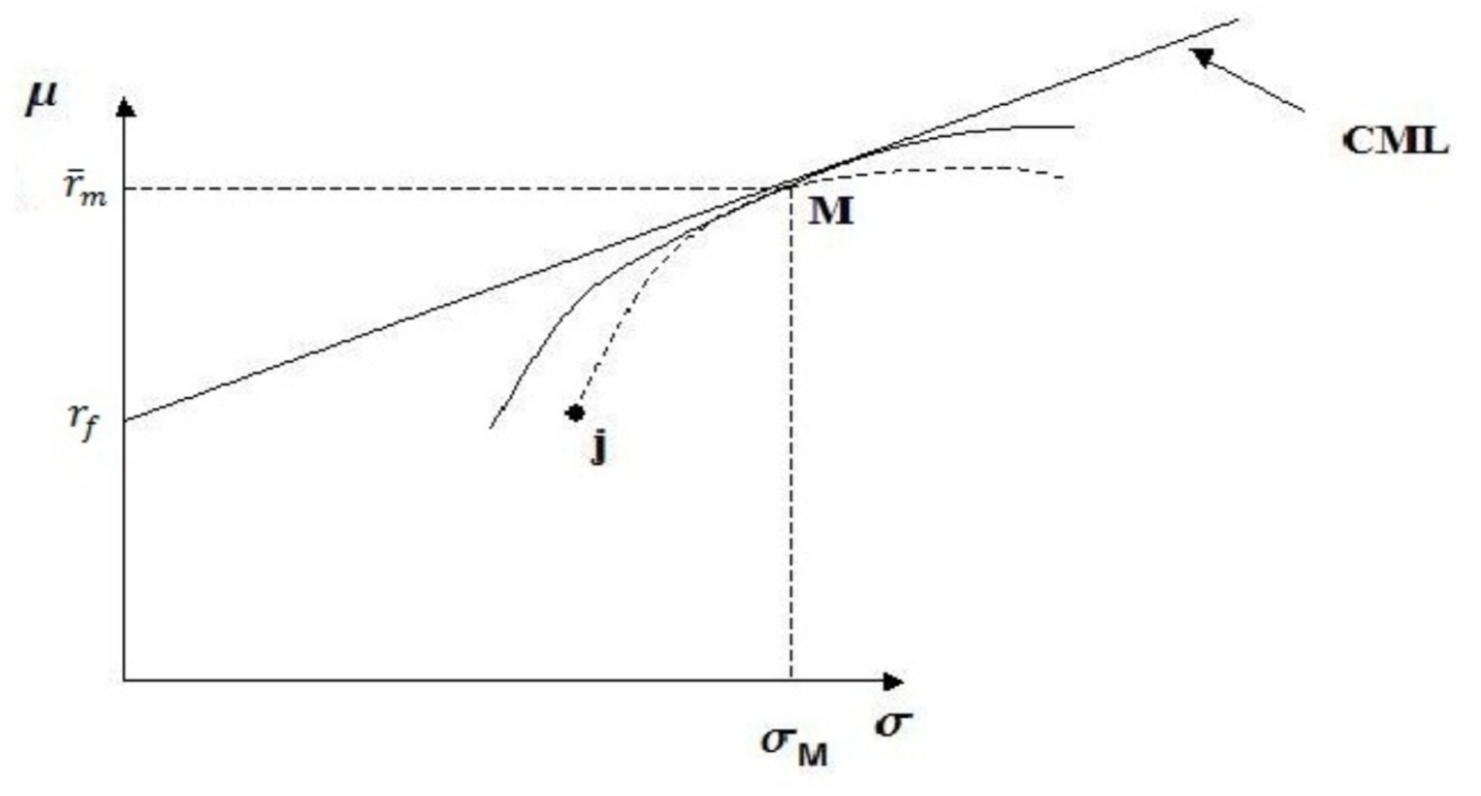
Formal Two Fund Separation Theorem

- If every agent's risk tolerance is linear with common slope γ, then date-1 consumption plans at any Pareto optimal allocation lie in the span of the risk-free payoff and the aggregate endowment

CAPM discussion (L5S31)

- CAPM makes no connection between financial markets and the real economy
- It assumes that supply = demand and markets are in equilibrium
- Used in practice, it equates historical asset returns with future asset returns (survivorship bias in efficient portfolio)
- It identifies only one source of risk: the market

- Formal assumptions:
- Each investor maximizes a mean-variance utility
- All investors share a common time horizon and homogeneous beliefs about expected returns and variances of existing assets
- There exists a risk-free asset and short sales of the risk-free asset are allowed
- The endowments of all agents are traded



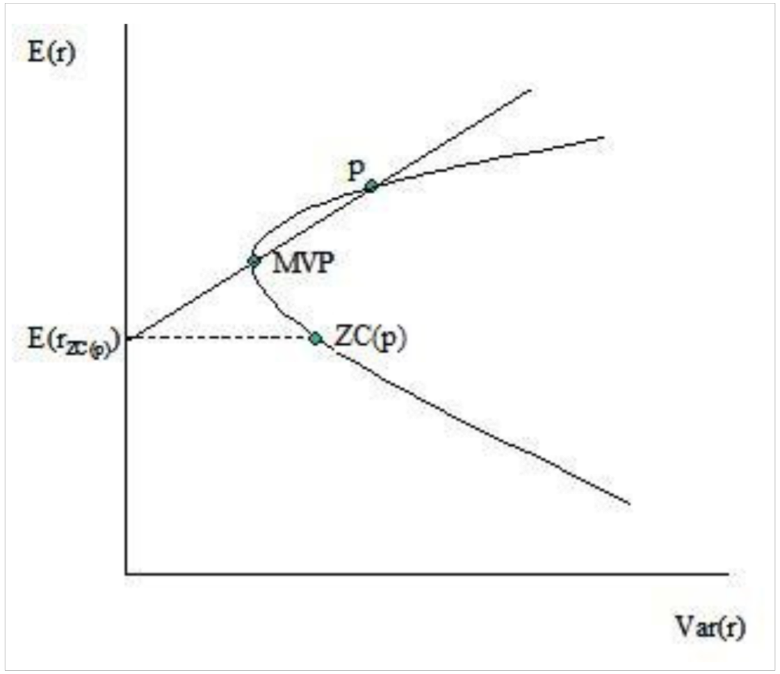
The Security Market Line SML (L5S36)

- Imagine a portfolio of an asset j and the market portfolio M
- As we change the weight a of asset j, we move along the dotted line, the SML
- At the point where our new portfolio has the same risk and return as the market, the CML and SML have the same slope
- We know this slope as the Sharpe ratio $\frac{r_M - r_f}{\sigma_M}$

SML II

- Using the chain rule, we find that the slope of the portfolio return with respect to the weight a of asset j is: $r_M - r_j$
- Thus:
- $r_j = (r_M - r_f)\frac{\sigma_M}{\sigma_M^2}$
- Or the more familiar version:
- $r_j = \frac{r_M - r_f}{\sigma_M}\ \beta_j \sigma_M$

- $\beta_j = \frac{\sigma_j}{\sigma_M^2}$describes the amount of market risk included in j
- $\beta_j \sigma_M$ is the systematic, undiversifiable risk
- A more direct way of showing the relationship of j and the market is:
- $r_j = \frac{r_M - r_f}{\sigma_M}\ \rho_{jM} \sigma_j$



From CAPM to zero-beta CAPM (L5S44)

- CAPM assumes that expected returns and variance-covariance matrices are given
- In reality, they are not
- We can still price assets under equilibrium arguments
- In equilibrium, investors will hold efficient portfolios and the market portfolio will be on the efficient frontier
- For all efficient portfolios there will be a non efficient sister portfolio ZC(p) on the frontier line, which has zero covariance with p

Zero-Beta-CAPM II

- The return of an asset j is then
- $\mathbb{E}(r_j) = \mathbb{E}(r_{ZC(M)}) + \beta_{Mj}[\mathbb{E}(r_M) - \mathbb{E}(r_{ZC(M)})]$
- It is easier to think of $ZC(M)$ as an ‘anti market’ or ‘safe haven’ portfolio (Gold, etc.)
- Every asset then lies on a line between the save haven and the market, with β_{Mj} indicating how close it is to the market
- It is called ‘Zero-Beta’ CAPM because the beta between market and ‘save haven’ ZC(m) is zero

Why is the yield curve upwards sloping? (L6S3)

Expectation hypothesis

- Current long term yields are current short term yields plus expected future short term yields
- Assuming: Long / short term bonds are perfect substitutes, free of default risk and agents are *risk neutral*
- Investors expect short term yields to increase
- Does not hold if there is uncertainty about future yields

Liquidity preference

- Assumes agents are *risk averse*
- They prefer liquid assets
- Long term, illiquid assets have to pay a liquidity premium

Market segmentation

- Different buyers have different preferred holding periods
- Thus, there are different, although connected, markets for long / short term bonds

Constructing term structures of interest rates

- ‘Term structure’ = Prices and effective interest rates of the different bonds paying different coupon rates over different periods
- Main idea: The interest rate in every year is the same for all bonds
- For a bond with price p and coupon c
- $p = \frac{c}{r_0} + \frac{c}{r_1} + \frac{c}{r_2}$ where r_i is the interest rate at time i
- Usually, you are given a series of bonds with their prices, calculate the first year rate from the 1 year running one, and so on

Zero-Coupon Bond (L6S40)

- ZCB's pay $1 at maturity and no coupon before
- The short rate is the price of a ZCB that matures instantly
- A forward rate from T1 to T2 can be 'locked in' by selling one bond expiring at T2 and use the proceeds to buy a T2 bond

Espinoza et al. (L6S48)

- Why is the yield curve upward sloping? Our professor and his ex. PhD student have their own opinions
- *Even in absence of aggregate uncertainty, future states of nature with higher spot interest rates have higher state prices. This generates a liquidity-term premium.*
- Their model is a two period economy with one borrower Alice and one lender Bob
- Alice is endowed with no commodities in t=0 but receives some commodity in every state at t=0, although possibly different amounts
- Bob is endowed with commodities in t=0, but will receive no commodities in the future
- Thus, Bob will make a loan to the borrower, so that the lender will gain commodities in the future and the lender can consume in t=0
- The lending process works through arrow securities, Alice sells arrow securities to Bob and pays if certain state occur

Generalized Equilibrium Approach

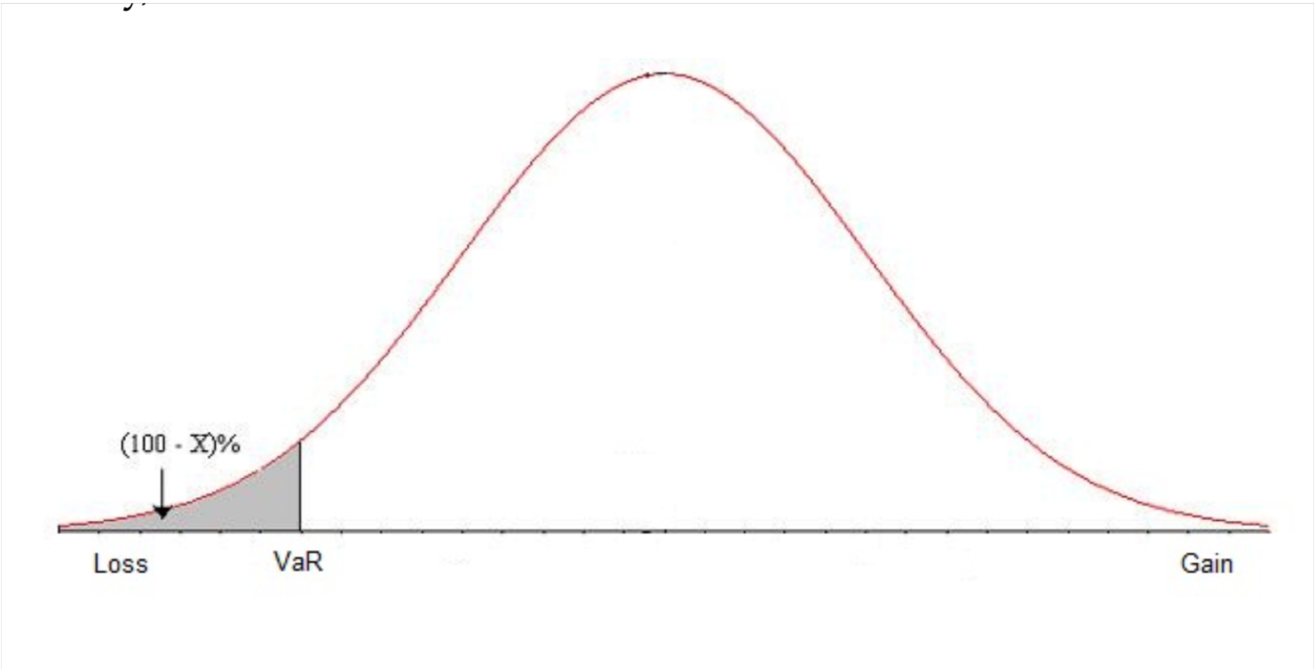
- Both maximize utility as a function of current and future consumption
- Both agents are constraint, as consumption + spending on buying assets ≤ gains from assets + gains from selling assets + endowments
- To find the optimal points, we need to set the derivative of the utility function with respect to decision variables (amnt bought, sold, etc.) to zero
- Making use of the equilibrium argument, we set total amounts bought + sold in the economy to zero (Selling is negative buying)
- Using these two equation sets, we can solve for the decision variables
- For the solution to be Pareto optimal, utility derivatives should be equal for all agents

VaR and Basel (L7S9)

- Credit institutions are required to keep enough equity to withstand a worst case scenario of their risky bets going bad over a period of time
- Basel I requires a share of ‘Risk Weighted Assets (RWA)’, which are loans made times a risk weight of that loan
- Basel I Amendment based requirements for trading book on Value at Risk (VaR)
- Basel II and 2.5 moved more measures from RWA to VaR

What *is* VaR

- What loss level are we X% confident of not exceeding in N business days?
- Most common metric is 10 day, 99% VaR
- Simplified estimate assuming independent daily outcomes
- 10 day VaR = √10 * 1day VaR



Estimating default probability from credit spread

- A corporate (zero coupon) bond has a higher price Pc than a risk free treasury bond Pf
- The discount factor d is the expected default probability over the life time of the bond
- $\frac{p_f - p_c}{p_f} = d$
- Probability of default times 1 - recovery rate (loss in event of default) equals expected loss
- Various credit ratings are based on this idea, since they assume ZCB's, they ignore that claims in event of default are higher (include coupon)

Rational Expectation Equilibrium (L8S9)

- If differently informed agent don’t learn from prices, they will hold different beliefs about prices forever and will not trade
- For markets to clear, agents thus need to learn from prices
- A price system has two roles now:
- > Determining an individual’s budget constraint, as in a competitive equilibrium
- > Conveying information
- An REE is similar to the competitive equilibrium studied in other lectures with the extension of differential information

Pricing with symmetric information

- Two traders with initial endowment x of the risky asset can choose to sell and buy a risk free asset instead
- Equilibrium condition means that all risky asset sold must find a buyer so the total amount held stays constant
- The only determinant of price is expected value v & the riskiness (variance of values) of the asset and risk aversion r of agents
- $p = \mathbb{E}(v) - r(x)Var(v)$
- r is the aggregate risk aversion of agents
- The second term is a risk premium

Pricing with **asymmetric** information

- Assuming only two traders:
- We assume that an assets future value is driven by signals s plus some unexplained random error
- **Even if an agent has incomplete information two start with, by observing the other trader, it can infer the signals the other trader has. Traders thus converge on the information they have**
- $p = s - r(x)Var(\varepsilon)$
- Now, the risk premium is determined by risk aversion, the amount of assets at stake and the variance of the unexplained error

Noisy REE (L8S23)

- With only (semi) informed and uninformed traders, it is always possible to infer the other's signals and valuations from their trading behavior (prices are fully revealing)
- Thus, nobody would bother to collect private information
- But then, how would information get into the market in the first place? **Informationally efficient markets are impossible**
- For prices to not be revealing, there have to be random noise traders which obfuscate the relationship between price and signal
- This allows informed traders to have an advantage as their information is not revealed, thus incentivizing information collection

Noisy REE II

- From analyzing optimization objectives:
- Returns from signal divided by returns from noise equal share of informed traders divided by risk aversion of informed traders time unexplained variance
- $\frac{\beta}{\gamma} = \frac{\lambda}{r_I V_\epsilon}$
- Thus, many, aggressive (low risk aversion), informed traders lead to more revealing markets

But is it Pareto Optimal? (L8S29)

- Need to distinguish between ex ante (outcome not revealed and no private signals), interim (agents receive private signals) and ex post (outcome is revealed)
- The ex post REE is the same as a competitive equilibrium and thus PO
- If an agent takes advantage of private information at interim, it will leave the other side of the trade worse off, thus, **if an REE is ex ante PO, it is interim PO & ex post PO**

Kyle Model (L8S32)

- Kyle departs from REE, agents are aware of their influence on price and might deliberately manipulate it
- Then, even with zero noise, prices are only partially revealing
- In the Kyle model:
  - There is one risky and one riskless asset
  - There is a single informed, risk neutral insider, maximizing expected wealth
  - The insider knows the outcome of random asset return v
  - There are multiple noise traders
  - The trading format is a batch auction, orders are submitted without knowledge at which price they will be executed

Kyle II

- If the aggregate order flow is large, the market maker rationally infers that the insider has good news, and so the market maker sets a higher price
- λ is the amount the market maker raises prices if order flow goes up by one unit, an inverse market depth
- $p = \lambda(\theta + x)$, θ = insider trade, x = noise trade
- The insider takes this into account. The greater the price impact of his trading the less aggressively he trades
- Because of the insiders considerations:
- $\lambda = 1/2\sqrt{\frac{var(v)}{var(x)}}$
- If var(x) is lower, noise traders contribute less to the order flow, so contribution flow and thus information leakage increases